# Thermodynamic Wind Turbine Model Addendum


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Larry Mansberger
Mansberger Aircraft Inc.
758 Aviator Dr.
Fort Worth, Texas 76179

**A mathematical model for a thermodynamically active wind turbine was presented in the author's previous papers, "*A Fundamental Revision of Wind Turbine Design Theory*" and "*Corrected Momentum and Energy Equations Disprove Betz's Limit.*" This paper expands on their fundamental concept, the intent of which was to correct the conventional wind turbine momentum theory which diverges from real flow and to correct the improperly used energy equation which ignores thermal energy and infers false limitations on wind turbine performance. The basis for this new theory is the corrected momentum and energy equations, along with the fundamental realization that the product of axial force normal to the turbine disc times the axial velocity should not be directly equated to the power extraction of a rotating turbine. In this paper it is mathematically shown how this term is representative of the power available but manifests itself as a shift in the internal energy of the flow field and is not necessarily the source of the energy extracted. This concept is contrary to conventional theories. A new relationship is derived for efficiency of the wind turbine which determines how much of this available power can be extracted by the system and it is shown how to increase both the available energy and the energy extracted.**

## Nomenclature

| | | |
|---|---|---|
| $A$ | = | area |
| $a$ | = | axial induction factor, $a = (V_1\text{-}V_2)/ V_1 = (1\text{-} a_i)$ |
| $a_i$ | = | inflow velocity ratio, $a_i = V_2/ V_1 = (1\text{-}a)$ |
| $a_s$ | = | spinner acceleration factor, $a_s = V_{2.5}/ V_2$ |
| $b$ | = | axial slipstream factor, $b = (V_1\text{-}V_6)/ V_1$ |
| $b_i$ | = | outflow velocity ratio, $b_i = V_6/V_1 = (1\text{-}b)$ |
| $Cp$ | = | power coefficient |
| $c_p$ | = | specific heat at constant pressure |
| $c_v$ | = | specific heat at constant volume |
| $e$ | = | energy per unit mass |
| $D$ | = | drag force acting on airfoil parallel to relative flow |
| $h$ | = | enthalpy per unit mass |
| $K$ | = | ratio of change in enthalpy to $ke$ |
| $k$ | = | specific heat ratio $c_p/c_v$ |
| $F$ | = | force |
| $ke$ | = | kinetic energy per unit mass |
| $L$ | = | lift force acting on airfoil perpendicular to relative flow |
| $M$ | = | momentum |
| $\dot{m}$ | = | mass flow $\dot{m} = \rho VA$ |
| $P$ | = | power |
| $p$ | = | pressure |
| $q$ | = | dynamic pressure $q=\rho V^2/2$ |
| $R$ | = | maximum radius or gas constant in equation of state |
| $r$ | = | local or relative radius of blade element |
| $T$ | = | thrust or temperature |
| $V$ | = | velocity, with subscript defining location |
| $W$ | = | work |
| $\alpha$ | = | angle of attack of an airfoil |
| $\delta$ | = | angle between the resultant and normal blade forces |
| $\theta$ | = | pitch angle of blade airfoil |
| $\phi$ | = | relative flow angle from rotor disc |
| $\lambda$ | = | tip speed ratio, $\lambda = \Omega R/V_1$, $\lambda_R$ may also be used |
| $\lambda_r$ | = | local speed ratio, $\lambda_r = \Omega r/V_1$ |
| $\lambda_s$ | = | slipstream speed ratio, $\lambda_s = \omega r/V_1$ |
| $\lambda_S$ | = | slipstream outer speed ratio, $\lambda_S = \omega R/V_1$ |
| $\rho$ | = | density |
| $\tau$ | = | torque |
| $\Omega$ | = | angular velocity of turbine |
| $\omega$ | = | angular velocity of slipstream |

**Subscripts**

| | | |
|---|---|---|
| *rel* | = | relative |
| n | = | normal to turbine disc |
| r | = | with respect to an annular element located at radius $r$ |
| s | = | shifting with respect to energy or spinner with respect to radius |
| $\theta$ | = | with respect to the direction of rotation, tangentially |
| $R$ | = | resultant when referring to force vectors |

## I. Introduction

In this paper I will elaborate on the thermodynamic wind turbine model. The basis for this new theory is the corrected forms of the momentum and energy equations along with the new fundamental concept that the axial force normal to the turbine disc, the thrust force times velocity, should not be equated to the power extraction of a rotating turbine. In this paper it will be demonstrated how this term $F_n V_2$ is representative not of power output but of power available that manifests itself as a shift in the internal energy of the flow but is not the source of the energy extracted. The implication from this leads to the derivation of an equation for the efficiency of a wind turbine which determines how much of the available energy can be extracted. New equations are developed for the power coefficient *Cp* as a function of this new efficiency factor and as a function of a new accelerated flow factor. It will further be shown how to increase both the available energy and the energy extracted. New equations for optimization of wind turbine performance are derived and suggest that a return to multi-bladed designs with constrained accelerated airflow could achieve performance gains previously thought to be unattainable. This is made possible with the understanding of the thermodynamically active wind turbine. This paper does not stand alone but is based on and requires referencing to one or the other of the previous papers, *A Fundamental Revision of Wind Turbine Design Theory*[1] and *Corrected Momentum and Energy Equations Disprove Betz's Limit.*[2]

## II. The Thermodynamic Wind Turbine Model Continued

In the previous papers a realistic hypothetical case study was carried out for a small scale wind turbine. The study was completed for both a free-spinning wind turbine with no energy extraction and for a turbine with a power coefficient, *Cp* equal to 0.40. The parameter details and the results of the analysis are repeated on the following page along with Fig. 1 which identifies the station positions.

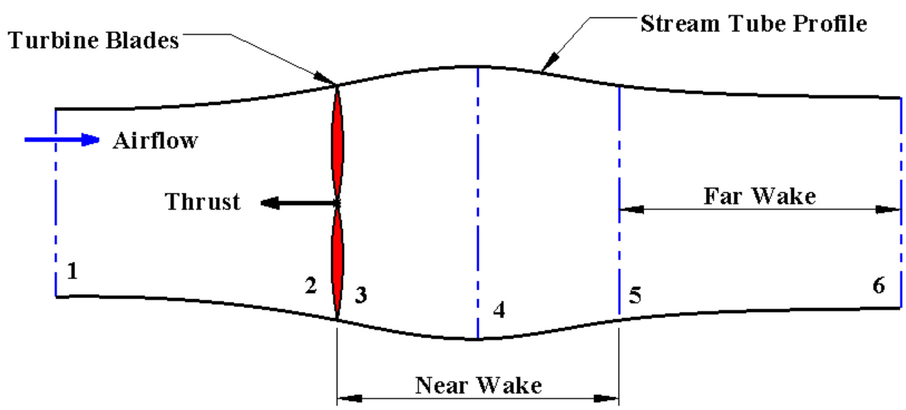


**Figure 1. Station Positions**

Summary of Sample Case Study Results (*Corrected revision 2015-10-18 LM)*

Given Parameters

| | |
|---|---|
| Diameter 6 ft, R = 3 ft | $V_\infty$ = 30 ft/s |
| Area = 28.27 $ft^2$ | $p_\infty$ = 2116.2 lb/$ft^2$ |
| $Cp$ = 0.40 | $\rho_\infty$ = 0.002378 slug/$ft^3$ |
| $a = 1/3$, $a_i=2/3$ | $\lambda_R = 6$ |

Constants

| | |
|---|---|
| $c_p$ = 6007.79 $ft^2/s^2/°R$ | $R = (c_p - c_v)$ = 1716.51 $ft^2/s^2/°R$ |
| $c_v$ = 4291.28 $ft^2/s^2/°R$ | $k = c_p / c_v$ = 1.40000 |

Calculated Results

| | |
|---|---|
| $\dot{m}$ = 1.334478 slug/s | $q$ = 1.0701 lb/$ft^2$ |
| $e_{out}$ = 270 $ft^2/s^2$ | $C_T$ = 0.911338 |

$T_1$ = 518.440°$R$
$p_1$ = 2116.20 lb/$ft^2$
$\rho_1$ = 0.002378 slug/$ft^3$
$V_1$ = 30 ft/s
$h_{01}$ = 3115128.6 $ft^2/s^2$

$T_2$ = 518.4816°$R$
$p_2$ = 2116.7944 lb/$ft^2$
$\rho_2$ = 0.00237848 slug/$ft^3$
$V_2$ = 20.00 ft/s
$h_{02}$ = 3115128.6 $ft^2/s^2$

| Energy Extracting | Free-Spinning |
|---|---|
| $T_3$ = 518.4363°$R$ | $T_3$ = 518.4816°$R$ |
| $p_3$ = 2115.8192 lb/$ft^2$ | $p_3$ = 2115.8192 lb/$ft^2$ |
| $\rho_3$ = 0.00237759 slug/$ft^3$ | $\rho_3$ = 0.00237738 slug/$ft^3$ |
| $V_3$ = 20.00 ft/s | $V_3$ = 20.00 ft/s |
| $h_{03}$ = 3114858.6 $ft^2/s^2$ | $h_{03}$ = 3115128.6 $ft^2/s^2$ |

| Energy Extracting | Free-Spinning |
|---|---|
| $T_4$ = 518.4411°$R$ | $T_4$ = 518.4864°$R$ |
| $p_4$ = 2116.20 lb/$ft^2$ | $p_4$ = 2116.2 lb/$ft^2$ |
| $\rho_4$ = 0.00237800 slug/$ft^3$ | $\rho_4$ = 0.00237779 slug/$ft^3$ |
| $V_4$ = 18.5 ft/s | $V_4$ = 18.44 ft/s |
| $h_{04}$ = 3114858.6 $ft^2/s^2$ | $h_{04}$ = 3115128.6 $ft^2/s^2$ |

| Energy Extracting | Free-Spinning |
|---|---|
| $T_5$ = 518.4363°$R$ | $T_5$ = 518.4816°$R$ |
| $p_5$ = 2116.4386 lb/$ft^2$ | $p_5$ = 2116.4377 lb/$ft^2$ |
| $\rho_5$ = 0.00237829 slug/$ft^3$ | $\rho_5$ = 0.00237808slug/$ft^3$ |
| $V_5$ = 20.00 ft/s | $V_5$ = 20.00 ft/s |
| $h_{05}$ = 3114858.6 $ft^2/s^2$ | $h_{05}$ = 3115128.6 $ft^2/s^2$ |

| Energy Extracting | Free-Spinning |
|---|---|
| $T_6$ = 518.4196°$R$ | $T_6$ = 518.4650°$R$ |
| $p_6$ = 2116.2 lb/$ft^2$ | $p_6$ = 2116.2 lb/$ft^2$ |
| $\rho_6$ = 0.00237809 slug/$ft^3$ | $\rho_6$ = 0.00237789 slug/$ft^3$ |
| $V_6$ = 24.4949 ft/s | $V_6$ = 24.4949 ft/s |
| $h_{06}$ = 3114858.6 $ft^2/s^2$ | $h_{06}$ = 3115128.6 $ft^2/s^2$ |

Note that in the results, although the power coefficient *Cp* was defined as 0.40, the actual ratio of the reduction in the kinetic energy of the flow is only equal to 0.33. The additional 6.7% of relative power extracted is the result of the extraction of internal thermal energy as explained in the previous papers. The reduction in kinetic energy relative to the free stream can be calculated as follows:

$$\frac{V_1^2 - V_6^2}{V_1^2} = 1 - b_i^2 = 1 - a_i = a\,, \tag{1}$$

where $a_i$ and $b_i$ are the inflow and outflow velocity ratios as previously defined[1, 2] and $a$ is the axial induction factor. Note that by this definition and for non-accelerated flow, the ratio of the reduction in kinetic energy is also equal to the ratio in reduction in mass flow which is equal to the axial induction factor and none of these terms is necessarily a function of the power extracted. As can be seen in the previous results, whether free-spinning or power-generating, both cases have the same reduction in kinetic energy of 33%. Although neither axial induction factor nor inflow velocity ratio can be independently related to power extraction, they can be related directly to turbine thrust in the following laminar wake momentum equation.[1,2]

$$a_i = b_i^2 = (1 - 0.5C_T)^{2/3} \tag{2}$$

Summarizing, the reduction in kinetic energy, the turbine wake profile and the mass flow rate all are strictly functions of the turbine thrust but not of the energy extraction. The foundation of conventional theory relies heavily on the misconception that the product of the turbine thrust times the velocity through the turbine $F_nV_2$ can be equated directly to the power extracted. This appears to work in conventional theory; in actuality it works only as an estimation for a very narrow region of flow and it causes a misinterpretation of flow parameters for most cases.

Let us look at this closer with the new theory in the Sample Case Study. The pressure drop across the turbine was shown in Eq. (58) of Ref. 2 to be equal to $-qC_T$. If we solve for $C_T$ from Eq. (2), the result is

$$\Delta p_{2,3} = 2q\left(1 - a_i^{3/2}\right). \tag{3}$$

For the conditions in the Sample Case, $-qC_T$ yields (1.07)(-0.9113)= -0.9752 lb/ft$^2$. We can now solve

$$F_n = \Delta p_{2,3} A_2\,. \tag{4}$$

$F_n$ = (-0.9752 lb/ft$^2$)(28.27 ft$^2$) = 27.57 lb.

We can now calculate $F_nV_2$ = (27.57 lb)(20.0 ft/s) = 551.4 ftlb/s. Dividing this by the mass flow rate of 1.345 slug/s equals 410.0 ft$^2$/s$^2$ per unit mass flow. The energy extracted was previously calculated at 270 ft$^2$/s$^2$ per unit mass flow. This is obviously nowhere close to the work being done by the rotor in the axial direction and the difference is significantly higher than any losses due to drag. So where is this axial work going to and what is its effect?

## III. Internal Energy Shift within the Turbine Flow Field

In order to answer the above question I proceed with a detailed analysis of the movement or shifting of the internal energy within the turbine flow field. I define the internal energy shift $e_s$ as the absolute value of the quantity of energy which moves from kinetic energy to thermal energy or vice versa as it moves from one station to another.

$$e_s = \left|c_p(T_i - T_{i+1})\right| = |0.5(V_i^2 - V_{i+1}^2)| \tag{5}$$

The theorem here is that the total energy shift that occurs, which is equal to the sum of the absolute values of the individual station shifts, will equate to the axial term $F_nV_2$ as shown in Eq. (6).

$$\Sigma\, e_s = \frac{F_nV_2}{\dot{m}}\,. \tag{6}$$

Four distinct regions of flow were defined in the previous papers; stations (1-2), (3-4), (4-5), and (5-6). Using the results summarized on page 4, we will calculate $e_s$ for each region of flow.

$$e_{s(1-2)} = |0.5(V_1^2 - V_2^2)| = |0.5(30^2\text{-}20^2)| = 250.0 \text{ ft}^2/\text{s}^2$$

$$e_{s(3-4)} = |0.5(V_3^2 - V_4^2)| = |0.5(20^2\text{-}18.5^2)| = 28.88 \text{ ft}^2/\text{s}^2$$

$$e_{s(4-5)} = |0.5(V_4^2 - V_5^2)| = |0.5(18.5^2\text{-}20^2)| = 28.88 \text{ ft}^2/\text{s}^2$$

$$e_{s(5-6)} = |0.5(V_5^2 - V_6^2)| = |0.5(20^2\text{-}24.49^2)| = 100.0 \text{ ft}^2/\text{s}^2$$

Summing the results gives us $\Sigma\, e_s$ = 407.76 ft$^2$/s$^2$ which is within one percent of 410.0 ft$^2$/s$^2$ the value for $F_n V_2/\dot{m}$ which was calculated at the turbine disc. This result supports the theorem which I put forth.

The above calculations demonstrate that the axial work occurring at the turbine disc is clearly work which the airflow performs on itself with the effect of shifting the kinetic energy of the flow to internal energy and back depending on the flow field profile it has created. The work must be distributed throughout the pressure distribution within the flow field which alternately deaccelerates and reaccelerates the airstream altering its profile. The only work acting on the turbine which results in energy extraction is $\tau\Omega$, torque times angular velocity. This should make immediate sense when one considers that there is no axial movement of any turbine components and therefore no work is done to the turbine by the normal force. Examining a free-spinning turbine supports these conclusions. In common with the free-spinning turbine are the auto-gyrocopter and the auto-rotating helicopter. Both of these vehicles in gliding flight experience a steady force times velocity reacting with the free stream, but neither transfer any power back into their rotating shafts.

On a side note, the solution for the internal energy shift reveals an alternative method for estimating $V_4$ and the other thermodynamic parameters at station 4. This was defined as the location of maximum wake expansion within the region of non-isentropic flow. The original derivation for $V_4$ assumed $p_4 = p_\infty$ which may not be the case. But the kinetic energy shift between stations 3 and 5 must be split equally by station 4 implying $e_{s\,(3-4)} = e_{s\,(4-5)}$ . Therefore, we observe the following from above:

$$\frac{F_n V_2}{\dot{m}} = e_{s\,(1-2)} + e_{s\,(3-4)} + e_{s\,(4-5)} + e_{s\,(5-6)} \tag{7}$$

$$e_{s\,(3-4)} = \frac{1}{2}\left(\frac{F_n V_2}{\dot{m}} - e_{s\,(1-2)} - e_{s\,(5-6)}\right) = \frac{1}{2}(V_3^2 - V_4^2) \tag{8}$$

$$V_4 = \sqrt{V_3^2 - \frac{F_n V_2}{\dot{m}} + e_{s\,(1-2)} + e_{s\,(5-6)}}\,. \tag{9}$$

When we plug values for the Sample Case Study into Eq. (9) we get the following result:

$$V_4 = \sqrt{20^2 - 410 + 250 + 100} = 18.44 \text{ ft/s}\,.$$

This energy shift solution for $V_4$ is probably more accurate than the original thermodynamic solution which assumed $p_4 = p_\infty$. We can now return to the non-isentropic wake solution, from Eqs. (59) and (60) of Ref. 2, and using the new value for $V_4$ we can solve for the corrected value of $p_4$.

$$p_4 = \left(\frac{V_4}{V_2}\right)\left[\frac{1}{2}(p_3 + p_5 - 2p_\infty) - \dot{m}\,\frac{(V_4 - V_2)}{A_2}\right] + p_\infty \tag{10}$$

$$p_4 = \frac{18.44}{20.00}\left[\frac{1}{2}\big(2115.8192 + 2116.4386 - 2(2116.20)\big) - 1.3345\frac{(18.44 - 20.00)}{28.27}\right] + 2116.2 = 2116.2023$$

This solution agrees with the original assumption $p_4 \approx p_\infty$; although, the new solution for $V_4$ will be the more accurate. Ignored in the derivations was any account for the radial momentum of the flow which will probably cause slightly more expansion. These corrected values of $V_4$ and $p_4$ will have negligible results on the final thermodynamic parameters previously calculated. The corrected values for both cases are re-graphed in the following Fig. 2.

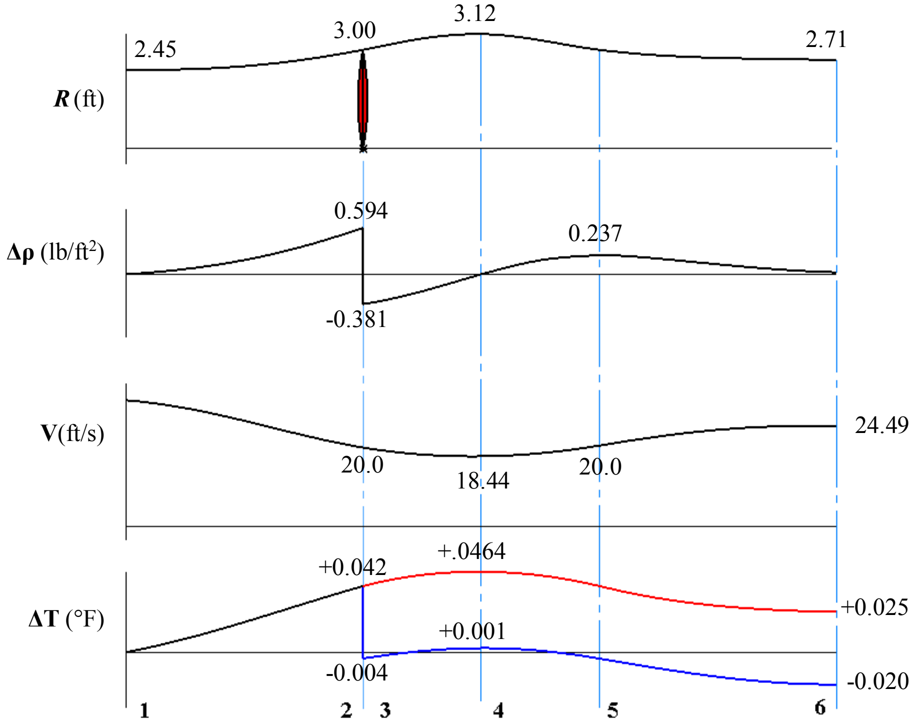


**Figure 2. Station Thermodynamic Parameters**
*Red, free-spinning; Blue, energy extracting*

## IV. The Force and Velocity Triangles Re-examined with Power

I will now return to blade element theory so we can better understand the mechanics which cause this paradox between the energy represented by $F_n V_2$ and actual energy extracted. Blade element theories start with a typical drawing depicting the forces and relative velocities acting on an airfoil section of the turbine blade. For details of conventional theory refer to Ref. 3 and Ref. 4. For now I will simplify the rotation into a linear blade actuation, such as considering a cascade of land sailers moving perpendicular to the wind. This would very closely represent the condition in an outer annular element of the multi-bladed wind turbine proposed by this theory. For this discussion refer to the following Fig. 3:

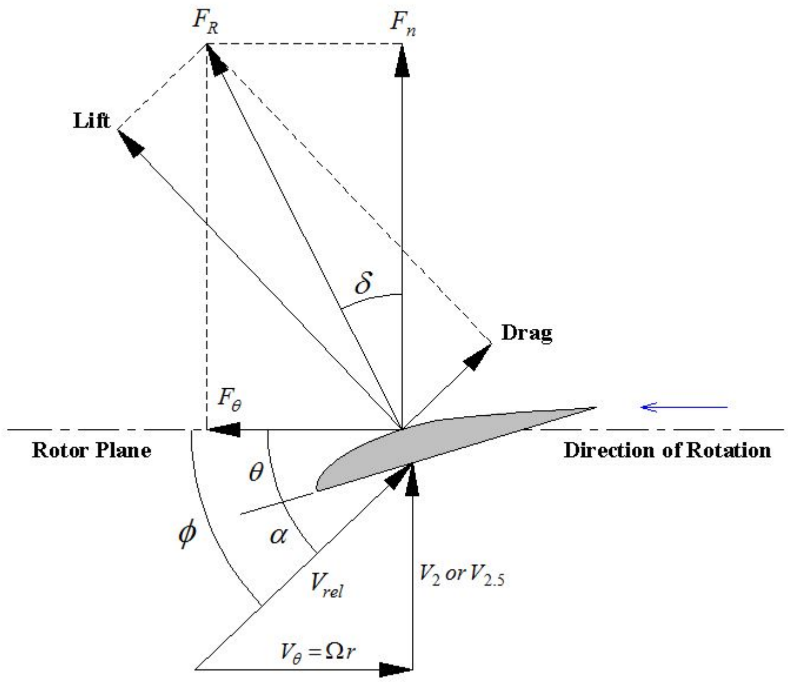


**Figure 3. Blade Element, Force and Velocity Triangles**

Note also in Fig. 3 I have defined a new angle δ, which is the angle between the resultant force created by the airfoil and the normal component of the force. Based on trigonometric relationships of Fig. 3, the following can be determined:

$$F_\theta = (D\cos\phi - L\sin\phi) \tag{11}$$

$$F_n = (L\cos\phi + D\sin\phi) \tag{12}$$

$$F_R = \sqrt{F_\theta^2 + F_n^2} = \sqrt{L^2 + D^2} \tag{13}$$

$$\sin\phi = \frac{V_2}{V_{rel}}\ , \qquad \cos\phi = \frac{V_\theta}{V_{rel}} = \frac{\Omega r}{V_{rel}}\ , \qquad V_{rel} = \sqrt{V_2^2 + V_\theta^2} \tag{14}$$

$$\delta = \phi - \arctan\left(\frac{D}{L}\right). \tag{15}$$

Power, $P$ or $\dot{W}$, can be defined as force times velocity yielding:

$$Power\ in\ = \dot{W}_{in} = F_n V_2 = F_R V_{rel} \cos\delta \sin\phi \tag{16}$$

$$Power\ out = \dot{W}_{out} = \tau\,\Omega = F_\theta V_\theta = F_R V_{rel} \sin\delta \cos\phi\ . \tag{17}$$

From this relationship it can be clearly seen that the term $F_n V_2$ is not equivalent to *power out*. Looking closer we can insert relationships Eqs. (11) - (15) from above and arrive at the following:

$$Power\ in = F_n V_2 = \sqrt{L^2 + D^2}\ \Omega r \sin\left(\phi - arctan\frac{D}{L}\right) \tag{18}$$

$$Power\ out = F_\theta V_\theta = \sqrt{L^2 + D^2}\ V_2 \cos\left(\phi - arctan\frac{D}{L}\right)\ . \tag{19}$$

Recalling sine and cosine addition formulas of

$$\sin(a - b) = \sin a \cos b - \cos a \sin b \tag{20}$$

and

$$\cos(a - b) = \cos a \cos b + \sin a \sin b \tag{21}$$

along with previous relationships, we can simplify the terms to

$$Power\ in = F_n V_2 = \frac{V_2 \Omega r L + V_2^2 D}{V_{rel}} \tag{22}$$

and

$$Power\ out = F_\theta V_\theta = \frac{V_2 \Omega r L - (\Omega r)^2 D}{V_{rel}}\ . \tag{23}$$

The temptation here is to ignore drag and come to the conclusion that $F_n V_2 = F_\theta V_\theta$. This was a paradox for me, at first, because it would not allow for or explain the conditions of the free-spinning turbine. We know if the turbine is free-spinning with no torque then $F_\theta V_\theta$ must = 0. Setting *Power out* = 0 for the free-spinning case gives us

$$\frac{V_2 \Omega r L - (\Omega r)^2 D}{V_{rel}} = 0 \tag{24}$$

$$V_2 \Omega r L = (\Omega r)^2 D \tag{25}$$

$$\frac{\lambda_r}{a_i} = \frac{L}{D}\ . \tag{26}$$

Inserting Eq. (25) into Eq. (22) for the free-spinning case yields:

$$F_n V_2 = \frac{((\Omega r)^2 + V_2^2) D}{V_{rel}} = D V_{rel} \neq F_\theta V_\theta = 0\ . \tag{27}$$

For the free-spinning case, drag is not necessarily equal to zero but is defined in relationship Eq. (26). The point here again is that $F_n V_2$ cannot be directly equated to *power out*.

The thermodynamic model shows $F_nV_2$ to be related to the internal energy shift, not *power out*, and yet there is a trigonometric connection between $F_nV_2$ and $F_\theta V_\theta$ which we are examining. But even in the absence of drag, $F_nV_2$ and $F_\theta V_\theta$ cannot be equated as can readily be seen for the free-spinning case:

$$Power\ in = F_nV_2 \approx \frac{V_2\Omega rL}{V_{rel}} \tag{28}$$

$$Power\ out = F_\theta V_\theta = 0\,. \tag{29}$$

## V. Understanding the Efficiency of a Wind Turbine

We can learn something from looking at a corollary between the efficiency of a propeller versus a wind turbine. In terms of horse power *hp,* a propeller's efficiency is often defined as

$$\eta_P = \frac{Power\ out}{Power\ in} = \frac{Thrust\ hp}{Shaft\ hp} = \frac{F_nV_2}{\tau\Omega}\,. \tag{30}$$

We could similarly define wind turbine efficiency as the inverse:

$$\eta_T = \frac{Power\ out}{Power\ in} = \frac{Shaft\ hp}{Thrust\ hp} = \frac{\tau\Omega}{F_nV_2}\,. \tag{31}$$

Using this definition for the efficiency of a wind turbine, we can further derive from Eqs. (22) and (23):

$$\eta_T = \frac{Power\ out}{Power\ in} = \frac{F_\theta V_\theta}{F_nV_2} = \frac{V_2\Omega rL - (\Omega r)^2 D}{V_2\Omega rL + V_2^2 D}\,. \tag{32}$$

Dividing Eq. (32) through by $V_1^2$ and $D$ yields:

$$\eta_T = \frac{a_i\lambda_r\left(\frac{L}{D}\right) - \lambda_r^2}{a_i\lambda_r\left(\frac{L}{D}\right) + a_i^2}\,. \tag{33}$$

Dividing Eq. (33) by $a_i^2$, factoring and further simplifying yields:

$$\eta_T = \frac{\frac{\lambda_r}{a_i}\left(\frac{L}{D} - \frac{\lambda_r}{a_i}\right)}{\frac{\lambda_r}{a_i}\left(\frac{L}{D}\right) + 1}\,. \tag{34}$$

Dividing Eq. (34) by $\lambda_r/a_i$ yields:

$$\eta_T = \frac{\frac{L}{D} - \frac{\lambda_r}{a_i}}{\frac{L}{D} + \frac{a_i}{\lambda_r}}\,. \tag{35}$$

I present this Eq. (35) as the fundamental equation for the efficiency of an annular element of a wind turbine for extracting available power from the wind, where the available power, synonymous with *power in*, is defined as $F_nV_2$. Notice that if $L/D = \lambda_r/a_i$, then the efficiency for extracting power goes to zero as is the case for the free-spinning turbine. Equation (35) is plotted with respect to $\lambda_r/a_i$ and L/D ratio in Fig. 4.

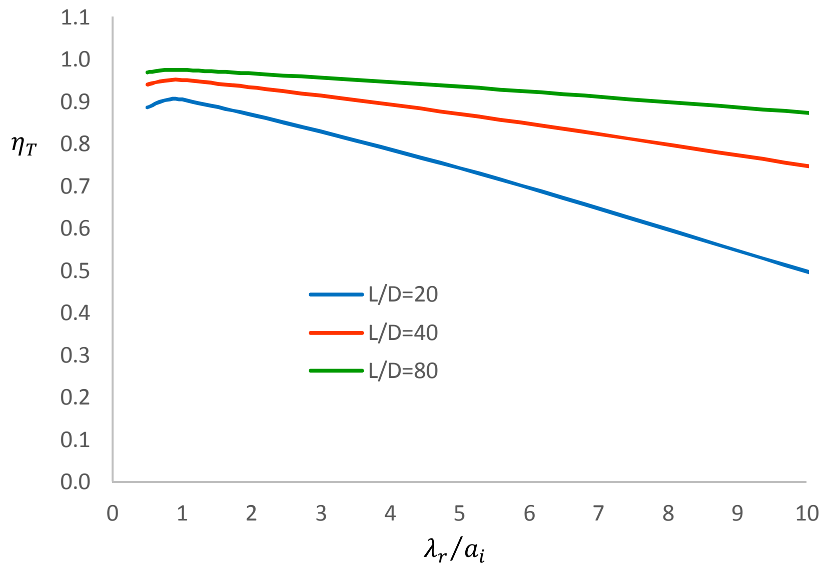


**Figure 4. Annular Blade Element Efficiency**

Also note from Fig. 4 and Eq. (35) that efficiency peaks at $\lambda_r/a_i \sim 1$ which implies $\tan\phi \sim 1$ or in other words the optimum flow angle would be approximately 45 degrees.

The above relationship for efficiency should not be confused with power coefficient $C_P$. By convention, $C_P$ is defined as the power extracted divided by the theoretical kinetic energy contained in the airflow.

$$C_P = \frac{Power\ out}{qV_1A_2} \tag{36}$$

From Eqs. (31) or (32),

$$Power\ out = \eta_T F_n V_2 \,. \tag{37}$$

So $C_P$ can be redefined as

$$C_P = \frac{\eta_T F_n V_2}{qV_1A_2} = \eta_T a_i C_T \,. \tag{38}$$

Furthermore, $C_T$ can be defined from Eq. (2) as,

$$C_T = 2\left(1 - a_i^{1.5}\right). \tag{39}$$

Inserting Eq. (39) into Eq. (38) gives us a new relationship for $C_P$ of

$$C_P = \eta_T 2a_i\left(1 - a_i^{1.5}\right). \tag{40}$$

Although efficiency is also a function of $\lambda_r/a_i$ , it is independent of power available and approaches unity for high L/D ratios. Therefore, we can approximately maximize the $C_P$ equation by taking the derivative with respect to $a_i$ and setting equal to zero while holding $\eta_T$ constant.

$$\frac{dC_P}{da_i} = \eta_T\left(2 - 5a_i^{1.5}\right) = 0 \tag{41}$$

Solving Eq. (41) yields an optimum of

$$a_i \approx 0.543 \ \text{and} \ C_{Pmax} \approx 0.651\eta_T \,. \tag{42}$$

This result is of major importance; that is, the realization that for unconstrained flow the optimum inflow velocity ratio should be $a_i \approx 0.543$ or an axial induction factor $a \approx 0.457$, not the $a$ value of one third that conventional design theory insists. This is apparent in Fig. 5 which compares the $C_P$ equations of the conventional Betz theory with the Mansberger Blade Element Theory at optimum efficiency for both.

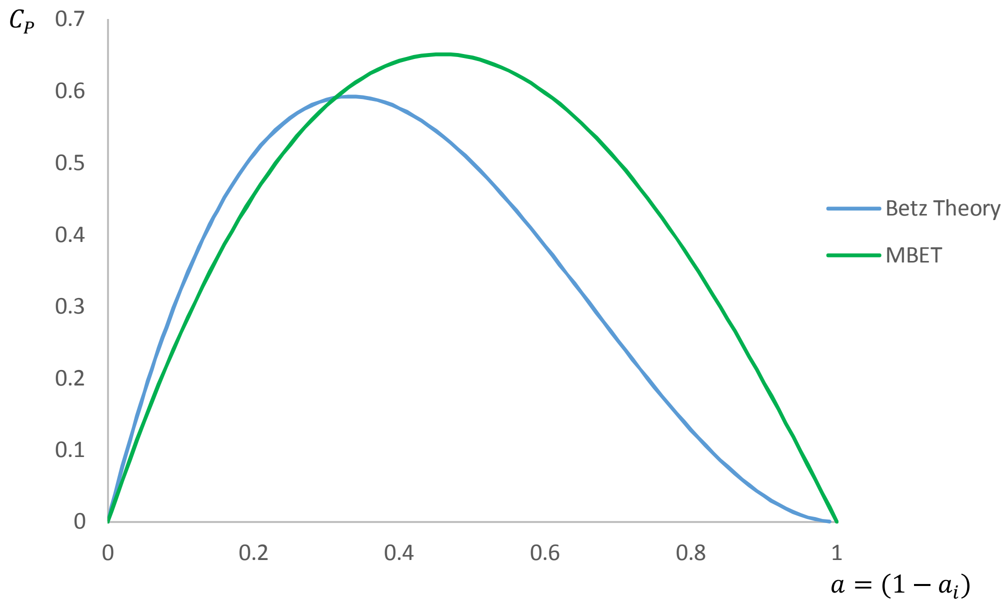


**Figure 5. *Cp* Theory Comparison**

These new values for $a_i$ and $a$ should be fairly accurate for most performance optimization. Notice by inserting the conventionally accepted optimum $a_i = 2/3$ into Eq. (40) yields a maximum value in approximate agreement with Betz's theory. In fact, the two theories almost parallel each other up to this point. This explains why for so long the conventional theory has been producing reasonable results and yet at the same time is limiting the full potential of wind turbines. But this is the region of flow where the old momentum theory diverges from reality and the new theory accurately predicts the flow. The result is not only a higher $C_P$ value, but as can be seen from Fig. 5 there is a much broader region of power generation capability just beyond the intersection where we have been designing the modern three bladed wind turbine. This is a region of the power equation just waiting to be taken advantage of by four and five bladed wind turbine designs. I will go as far as to predict in the very near future the major industry players will move to manufacturing five bladed designs to take advantage of this region of the new power curve. But as I will show, their efforts will still be underutilizing the total wind resource available.

## VI. Extracting More Power from the Wind

If our desire is to extract more thrust power from an aircraft powerplant, then looking at the equation $Thrust\ hp = \eta_P \cdot Shaft\ hp$, we must either increase the efficiency or increase the shaft horse power of the aircraft powerplant. If our desire is to extract more power from the wind, then looking at the equation $Shaft\ hp = \eta_T \cdot Thrust\ hp$ we must either increase the efficiency or the available thrust horse power reacting with turbine or in other words increase $F_n V_2$. We cannot increase $F_n$ without adversely affecting mass flow, but we can reduce the flow area increasing $V_{2.5}$ while holding a constant $F_n$. This brings us back to the accelerated flow concepts discussed in Ref. 1. By installing a large diameter spinner of radius $r_s$ and outer flow control ring it is demonstrated that $V_2$ can be increased to $V_{2.5}$. Equation (79) from Ref. 1 reads

$$V_{2.5} = \frac{V_2}{1 - \left(\frac{r_s}{R}\right)^2} \ . \tag{43}$$

We can alternately define a new term for spinner acceleration factor for constrained flow $a_s$ .

$$a_s = \frac{1}{1-\left(\frac{r_s}{R}\right)^2} \tag{44}$$

Equation (44) into Eq. (43) yields:

$$V_{2.5} = a_s V_2 \,. \tag{45}$$

This velocity increase simultaneously increases the kinetic energy and power available within the flow field, dropping the temperatures and increasing the efficiency of the system due to improved flow angles at the rotor. This new velocity term can be used to derive a new efficiency term for constrained flow with a spinner.

$$\eta_{Ts} = \frac{\frac{L}{D} - \frac{\lambda_r}{a_i a_s}}{\frac{L}{D} + \frac{a_i a_s}{\lambda_r}} \tag{46}$$

From Eq. (46) it can be shown that efficiency goes up with spinner ratio and spinner acceleration factor. Simultaneously, this makes the new power available term equal to $F_n V_{2.5}$ and we can conclude a new equation for power coefficient of:

$$C_P = \frac{\eta_{Ts} 2a_i\left(1-a_i^{1.5}\right)}{1-\left(\frac{r_s}{R}\right)^2} = \eta_{Ts} 2a_i a_s \left(1-a_i^{1.5}\right). \tag{47}$$

If we insert Eq. (44) and Eq. (46) into Eq. (47) the equation appears as:

$$C_P = \left(\frac{a_s\left(\frac{L}{D}\right) - \frac{\lambda_r}{a_i}}{\frac{L}{D} + \frac{a_s a_i}{\lambda_r}}\right) 2a_i\left(1-a_i^{1.5}\right). \tag{48}$$

Equations (47) and (48) are the newly derived equations for power coefficient which can be used for the design of the next generation of accelerated flow, thermodynamically active wind turbines.

Keep in mind that these equations do not account for the fact that $a_i$ and L/D are functions of both $\lambda_r$ and θ. None of these should actually be considered independent variables. That said, as an example for a spinner ratio of 0.65 and L/D=80, we can plot $C_P$ with respect to local speed ratio and inflow velocity ratio as shown in Fig. 6.

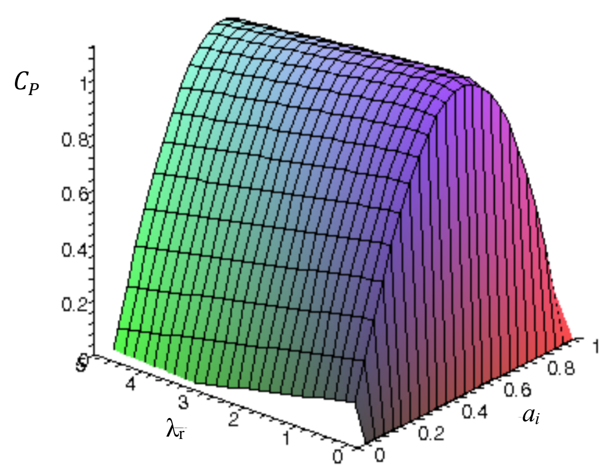


**Figure 6. *Cp* for Spinner ratio =0.65 and L/D=80**

The power coefficient in this example would not only exceed Betz's limit, but in addition shows the potential for $C_P > 1$, remembering it has been shown in this model that we can extract thermal energy as well as kinetic energy from the flow.

Yet another way to consider this whole concept is to imagine we inserted our wind turbine into an infinitely long uniform diameter duct. If the duct was held at a constant pressure, we would have a case similar to a hydro-turbine. Conservation of mass for the uniform duct maintains that the velocity of the flow never changes; therefore, the kinetic energy of the flow would remain constant. If the turbine is free-spinning within the pipe flow, this still creates a pressure drop that could be factored into a term typically defined as head loss. In the energy extracting case, the *power out* and temperature change would be dependent on the internal rotation. Within the duct the turbine can extract energy more efficiently and since the kinetic energy cannot change, the energy extraction can only come from within the internal thermal energy manifested as a pressure and temperature drop. The next generation of wind turbines will tap into this internal thermal energy in a similar manner.

## VII. Conclusion

The thermodynamic wind turbine model was further explored with new results derived. It was mathematically confirmed that the axial force normal to the turbine disc times the flow velocity $F_nV_2$ does not equate to power extracted. Rather, the term $F_nV_2$ is shown to be work acting on the flow stream itself, causing a shift back and forth within the total energy equation from kinetic energy to enthalpy along with associated changes in turbine wake profile. $F_nV_2$, or in the case of accelerated flow $F_nV_{2.5}$, can be related to the power available but these terms are not the source of the power extracted. The actual source of the power extracted will be a combination of enthalpy and kinetic energy terms from within the final total energy equation. The relationship between power available and power extracted is determined by an efficiency factor $\eta_T$ which is a function of inflow velocity ratio $a_i$, accelerated flow factor $a_s$, L/D, and local rotational speed ratio $\lambda_r$. New equations for power coefficients $Cp$ are derived based on these newly presented factors. For free flow around a turbine, the result is $C_P = \eta_T 2a_i(1 - a_i^{1.5})$, with the fundamental result that the optimum value for inflow velocity ratio is shown to be 0.543 or an axial induction factor of 0.457, not the conventionally accepted 1/3. This strongly suggests significant performance gains can be achieved from reconfiguring current designs along with the implementation of 4 and 5 bladed turbine configurations. For constrained flow it is shown that $C_P = \eta_{Ts} 2a_i a_s(1 - a_i^{1.5})$ and that $Cp$ values greater than one are theoretically possible. With this understanding, I predict an even greater potential for increasing power extraction will be found in constrained, accelerated-flow multi-bladed wind turbines.

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About the Author:

*Larry Mansberger holds a Bachelor of Science degree in Aerospace Engineering from the University of Texas with post graduate studies in engineering mechanics of composite materials and fluid dynamics. He has additional formal training in advanced composite design, fabrication and repair. He also holds FAA Certified Airframe and Powerplant Mechanic Licenses with FAA Inspector Authorizations. He has been flying since 1976 and is a current licensed private pilot with single engine land, seaplane and glider ratings. His 40 plus years of experience includes successfully designing, building and flying his own original design, all composite, multi engine, canard aircraft. Mansberger Aircraft has been in the business of providing specialty composite airframe services since 1985. This paper and other current endeavors focus on applying Mansberger Aircraft's unique skill set to wind energy research and product development. For more information visit* www.MansbergerAircraft.com.

*This paper is being self-published in the spirit of a free flow of information and ideas. Amazingly in the age of the internet, the author was surprised to find the research for this paper hindered by both stringent academia and restrictive professional journals. Peer review and discussion were found to be lacking in most of the work studied. In that regard, the author welcomes comments, corrections and discussion to be sent to* Larry@MansbergerAircraft.com.

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